Computer Algebra Systems Activity: Solving Diophantine Equations

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Topic: Solving Diophantine Equations

Ontario Expectations: (To be added when finalized by MOE.)

Notes to the Teacher:

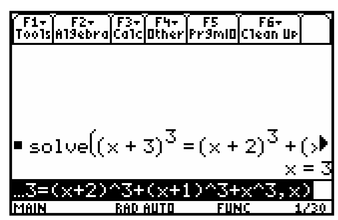
a) This activity is designed to use the CAS on the TI-89 calculator to enhance understanding and instruction. Other CAS systems may be used in place of the TI-89. All screen shots are from the TI-89.

b) The activity is presented in a Teacher Version, with all screen shots and solutions present, as well as a Student Version, which can be duplicated and handed out to students.

Teacher Version:

Introduction: A Diophantine Equation is one which has integer coefficients, and for which only integer solutions are desired. They are named after Diophantus, who live some time between 250 and 250 C.E.

1. Find four consecutive integers such that the cube of the largest equals the sum of the cubes of the other three. This problem leads to the Diophantine equation: $(x+3)^3 = (x+2)^3 + (x+1)^3 + x^3$. You can select **solve(** under the **F2** menu on your TI-89 to try to find a solution for this equation.



Is there a solution which consists of integers?

[Answer: yes. $6^3 = 5^3 + 4^3 + 3^3$.]

2. Some Math History: A famous Diophantine equation occurs in Fermat's Last Theorem. Fermat was considering the equation

$x^n + y^n = z^n$

If $n = 2$, the equation becomes the Pythagorean theorem, which has many integral solutions, such as 3, 4, and 5.

Find two other "Pythagorean triples", which are not multiples of 3, 4, and 5, or of each other.

[Answer (may vary): 5, 12, 13; 7, 24, 25.]

Fermat conjectured that there are no integral solutions when $n > 2$. As the story goes, he noted that he had found a proof of this conjecture, but the proof was not discovered among his papers when Fermat died in 1665. His theorem remained a conjecture until 1994, when Andrew Wiles published a proof.

3. Another Diophantine equation is Pell's equation

$x^2 - ny^2 = 1$

Consider the special case, when $n = 92$.

$x^2 - 92y^2 = 1$

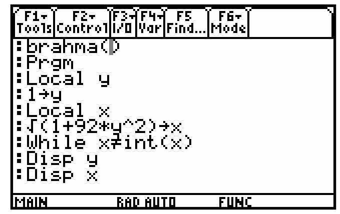
This is known as Brahmagupta's Problem. While many Diophantine equations have no solutions, this one does.

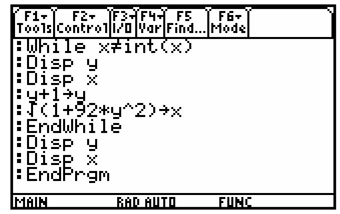
To find positive solutions, solve the equation for *x*, keeping the positive root.

[Answer: $x = \sqrt{1 + 92y^2}$]

By inspection, it is obvious that any real number substituted for *y* will result in a solution for the equation. However, the trick is to find an integral solution. You will use the programming power of your TI-89 to do this.

4. Select **7: Program Editor** from the **APPS** menu, and then **3: New**… . Scroll down to **Variable**, and type a name for your program, such as brahma. Press **ENTER** twice. Note that the TI-89 adds two brackets to your program name. These are used to pass values to a program, and will not be used in this program.





The program editor will open. Scroll to the end of the line **Prgm**, and press **ENTER**. Note that a blank line is inserted. You will use two **Local** variables Select **Local** from the **F4** menu and declare the variable ***y***. Store the value 1 in ***y***. Declare another local value ***x***, and store the expression derived above in ***x***. You have now initialized the values of the variables that you want to use.

You are looking for an integral value of ***y*** that makes the value of ***x*** an integer as well. You will use the "guess and check" method, starting from ***y*** = 1, and increasing the value of ***y*** by 1 until you find one that works. This is accomplished using a **While…Endwhile** loop from the **F2** menu. Use the **int()** function from the **CATALOG** to control the **While…Endwhile** loop. The **int()** function returns the largest integral value of a number which is less than or equal to the number.

If ***x*** = i**nt(*x*)**, then ***x*** must be an integer. When this condition is met, the loop ends. Note: the ≠ is found under the **MATH** menu, in **8: Test**. To keep track of progress, use the **Disp** command to show the values of ***y*** and ***x*** on the screen each time the program goes through its loop. If the solution has not been found, increase the value of ***y*** by 1, and recalculate the value of ***x***. Continue the loop. When the loop ends, display the final values of ***y*** and ***x***.

Check your program carefully to ensure that you have entered all lines using the correct syntax. Ensure that you understand the flow of the program, and what each line does.

Select **A: Home** from the **APPS** menu to return to the **Home** screen. Enter the name of your program, such as **brahma()** in the command line. Don't forget the brackets, even if you are not using them to pass variables. While the program is running, the Busy message will display in the lower right The program will run slowly enough that you can follow its progress on the calculator screen. What are the smallest integral solutions for ***x*** and ***y***?

[Answer: *x* = 1151 and *y* = 120.]

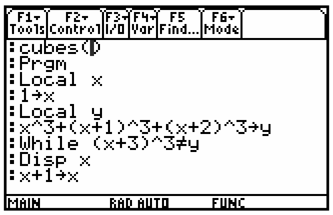
Note: If your program didn't work properly, and continued to run, you can stop it by pressing the **ON** key.

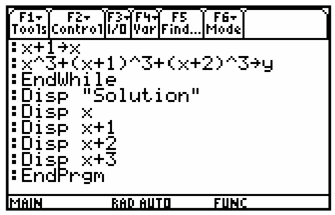
5. Adjust your program to find solutions of Pell's equation using other values, such as 8, 50 and 99.

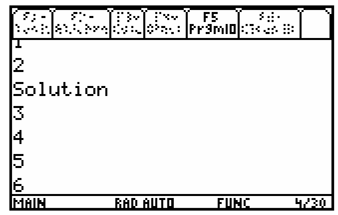
[Answer:

n	***x***	***y***
8	3	1
50	99	14
99	10	1

6. Write a program to solve the problem posed in part (1).







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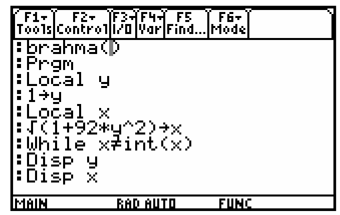

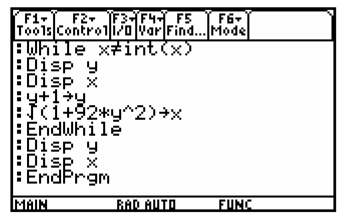

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